THE DEVELOPMENT OF AN ELECTRO-MECHANICAL ROOT-SOLVER

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INTRODUCTION

Throughout the fields of scientific research and engineering practice there is a great need for a fast and accurate method of obtaining the roots of an algebraic equation. The purpose of this thesis is to present the theory and design of a computer which is capable of yielding the real roots of an algebraic equation with constant coefficients both quickly and accurately. There are described in the literature several successful mechanical and electronic devices that perform this function.[1, 2] There is a continuing need for such a computer whose components are simple elements or simple groups of simple elements.

Complying with this requirement, the computer which is to be described contains only linear potentiometers, reversing switches, and isolating transformers and amplifiers. Its design is such that its range may be extended to accommodate algebraic equations of any degree by simply increasing the number of existing components. By applying several mathematical

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[1] H. C. Hart and I. Travis, "Mechanical Solution of Algebraic Equations," Journal of the Franklin Institute, 225: 63, 1938.

[2] A. H. Schooley, "An Electro-mechanical Method of Solving Equations," RCA Review, 3: 86, 1938.

theorems, all the real roots of an equation may be found with the computer. These roots may be very large, very small, or they may lie very close together. By the use of these theorems, the range of the roots which the computer can accommodate is unlimited.

THEORY OF THE METHOD OF SOLUTION FOR REAL ROOTS

An expression of the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n = 0 \qquad (1)$$

is the algebraic equation whose roots it is desired to find. At the present time it will be assumed that all the roots are real; however, a method will be described later by which complex roots may be obtained. In the computer, each term in the left member of Equation (1) is represented by the magnitude of an a-c voltage. By connecting these voltages in series and obtaining their sum, as indicated by a high-resistance voltmeter, the equation is completely represented. The voltmeter reading represents the right member of Equation (1) for some particular value of the independent variable. The real value of x that makes the voltmeter reading zero is a desired real root of the equation.

The method by which the various powers of x are obtained will now be considered. In the past, non-linear potentiometers have been used for this purpose.[3] These potentiometers must be specially manufactured with the required degree of taper so that the voltage output is the square, cube, etc., of the linear position of the movable arm. This requires that each term of the equation employ different potentiometers of different degrees of

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[3]Schooley, <u>loc</u>. <u>cit</u>.

non-linearity. This difficulty can be eliminated by the use of only linear potentiometers. By connecting two linear potentiometers in tandem, the output voltage is proportional to the square of the arm position, provided the two arms are ganged on a single shaft. A voltage proportional to the cube of the arm position is obtained by placing three potentiometers in tandem, etc. A voltage proportional to the nth power of the arm position can be obtained by using n potentiometers, as illustrated in Figure 1. If the voltage applied to n potentiometers represents the coefficient of the nth term of the equation, it can be seen from Figure 1 that the output voltage will represent the entire nth term. A pointer attached to the common shaft will indicate on a dial, calibrated linearly from 0 to 1.0, the particular value of x which is being substituted into the equation. By placing all the potentiometers to be used for obtaining the powers of x for each term on a single shaft, a rotation of this shaft will change the value of the independent variable in each term of the equation simultaneously. If separate and independently operated potentiometers are inserted prior to those connected in tandem, as shown in Figure 1, their outputs will supply the coefficients for each of the terms of the equation. By feeding each of these coefficient potentiometers from a common source of a-c supply, the absolute magnitude of the supply voltage will not affect the accuracy of the computer.

If any of the coefficients are negative, the reversing

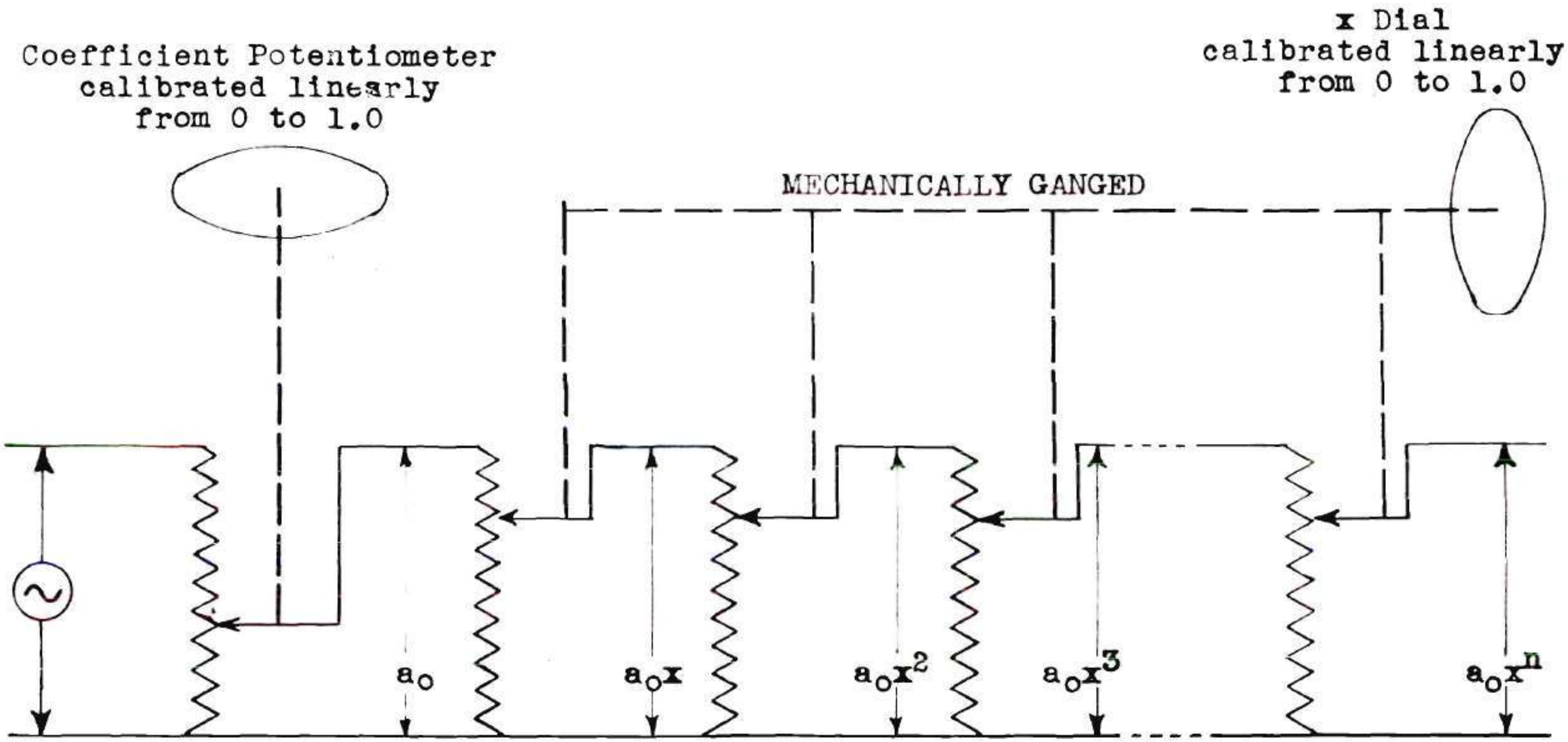


Figure 1

Circuit for obtaining a voltage, $a_o x^n$, which represents the nth term of an algebraic equation in x, and where a_o is the constant real coefficient of the nth term.

switch corresponding to that term is thrown to an opposite position, relative to the positions of the switches for the positive coefficients. The positions of the positive and negative coefficient switches are purely relative, since each term of the equation may be multiplied by -1 without affecting the value or sign of the roots. The outputs of each of the terms are connected in series with a high resistance voltmeter whose deflection will indicate the value of the algebraic sum of the terms of the equation for any particular value of the independent variable. Since it is desired to find the particular values of x which will give a zero reading on the voltmeter, the measuring device used should have as sensitive a scale as possible; this expedient will increase the accuracy of the computer.

In order to utilize the full accuracy of the computer, at least one coefficient of the equation must be equal to $\pm$ 1. This form is obtained by dividing each term by the coefficient with the largest absolute magnitude. After this operation, the potentiometers corresponding to the terms whose coefficients are unity are set at full scale. For the remaining terms whose coefficients are less than unity, the potentiometers are adjusted to a corresponding per cent of full scale. This method requires that each of the coefficient potentiometer dials be calibrated linearly from zero to unity. A complete circuit is shown in Figure 2.

The use of potentiometers to obtain the various powers

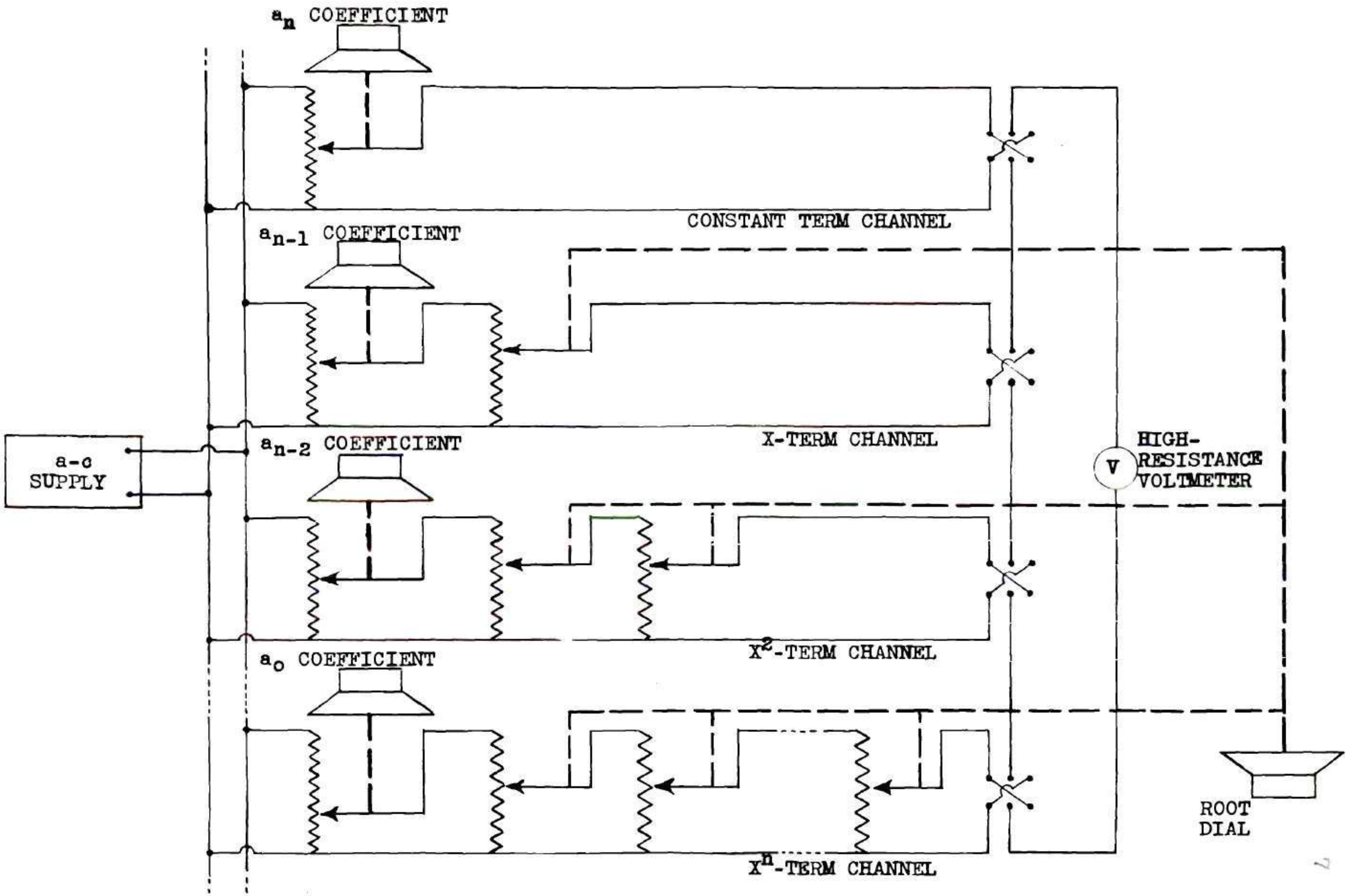


FIGURE 2

A Circuit for solving for the real roots of an algebraic equation with constant real coefficients.

of x limits the magnitude of the roots which can be obtained to unity. By transforming the given equation into an equation whose roots are k times the corresponding roots of the original equation, the range of the computer can be extended indefinitely. This procedure may be applied to an equation whose roots are very small. As an example, if a root has the value 0.004, the equation may be transformed into an equation whose root is now 0.4 by letting $k = 100$. The roots obtained from a transformed equation must be divided by k to obtain the root of the original equation. To transform an equation for application to the computer, k can always be chosen as some multiple of 10, making the theorem extremely simple to apply. If an equation has a negative root, it can likewise be obtained from the computer by letting $k = -1$ multiplied by the appropriate power of 10. This merely changes the sign of the coefficients of the odd powers of x. A statement of this theorem and an example are presented in Appendix II.

If it happens that the equation to be solved contains both real and imaginary roots, a preliminary examination of the equation is desirable. The computer will yield all the real roots if the procedure explained above is followed. It will also indicate the presence of complex roots, but will not yield their complex magnitudes. For this reason it is wise to determine the upper and lower limits of the real roots of the equation prior to attempting a solution. Additional tests may be applied to determine the lower and upper limits of the positive and negative roots, respectively, if desired. (See Appendix I.)

If the equation contains a multiple root, the computer will yield the magnitude of this root but not its degree of multiplicity. To obtain this information it is necessary to apply a theorem. Its application can, however, be deferred until the complete range of the real roots has been explored. If the number of roots indicated by the computer is less than the degree of the equation being solved, the theorem must be applied to determine whether one or more of these roots is repeated. (See Appendix III.)

After the multiple roots have been determined, if the number of real roots is still less than the degree of the equation, counting an m-fold root m times, the remaining roots will be complex and even in number. If there are only two complex roots, they can be found very easily from the depressed equation. The depressed equation is the quotient of the original equation divided by the product of the factors formed of the known roots.

Another useful relationship between the roots and the coefficients of an equation is that the algebraic sum of the real roots and the real parts of the complex roots is equal to the negative of the ratio a_1/a_0, where a_0 and a_1 are the coefficients of the x^n and x^{n-1} terms, respectively. This relationship may be used as a check on the results obtained from the computer.

The preceding theory can be summed up into ten steps which should be followed in the order given when attempting to solve an equation with the computer.

STEP 1 Find the upper and lower limits of the real roots. (See Appendix I.)

STEP 2 Divide each term of the equation by the coefficient with the largest absolute magnitude.

STEP 3 Set the coefficients of the equation obtained in Step 2 into the computer, along with the correct switch positions.

STEP 4 The range of the root dial is now from 0 to 1. Explore this region, at the same time noting the voltmeter indication. If the voltmeter reads zero, the reading of the root dial, at that time, is a real root of the equation.

STEP 5 Reverse the existing switch positions of the odd power terms and explore the region from 0 to -1 for additional roots.

STEP 6 Transform the original equation into one whose roots are the next sub-multiple of 10 of the roots of the original equation by letting $k = 1/10$. (See Appendix II.) The range of the root dial is now from 0 to 10. Explore this range for real roots.

STEP 7 Reverse the existing switch positions of the odd power terms and explore the region from $x = 0$ to $x = -10$.

STEP 8 Repeat Steps 6 and 7 for increasing sub-multiples of 10, by letting $k = 1/100$, $1/1000$, etc. The ranges of the root dial become 0 to ± 100, 0 to ± 1000, etc. The above process is continued until either of the two following conditions is satisfied:

(a) until the number of roots already obtained is equal to the degree of the original equation. If this be the case, the solution of the equation is complete.

(b) until the entire range between the upper and lower limits as defined by Step 1 has been explored.

STEP 9 If condition (b) but not (a) of the previous step has been satisfied, the remaining roots may be multiple roots or complex roots. If there remains an odd number of roots that have not been found, at least one will be a multiple root. Complex roots always occur in conjugate pairs if the coefficients of the equation are real. In either case test the equation for multiple roots. (See Appendix III.)

STEP 10 If no multiple roots are found and there remains an odd number of roots outstanding, it is highly probable that two roots lie very closely together. If they are so close that the computer cannot make a distinction between them, a transformation of the equation is required such that the roots of the new equation are farther apart than the roots of the original equation.

This transformation may be applied any number of times, but usually once will be sufficient. (See Appendix IV.)

To illustrate the above procedure, an example is presented here. Suppose it is desired to solve the following equation on the computer:

$$x^5 + 9.6x^4 + 18x^3 + 201.2x^2 + 1337x - 568.4 = 0 \qquad (2)$$

This equation was obtained by expanding the factored equation

$$(x - 0.4)(x + 7)^2(x - 2 - j5)(x - 2 + j5) = 0 \qquad (3)$$

STEP 1 An upper limit of the real roots is

$$1 + \frac{568.4}{1} = 569.4$$

A lower limit of the real roots is

$$- 1 - \frac{201.2}{1} = -202.2$$

A lower limit of the positive real roots is

$$\frac{568.4}{568.4 + 1337} = \frac{568.4}{1905.4} = 0.299$$

An upper limit to the negative real roots is

$$- \frac{568.4}{568.4 + 201.2} = - \frac{568.4}{769.6} = -0.74$$

The upper and lower limits of 569.4 and -202.2 are the only ones actually required at this time, but the others may be found if desired.

STEP 2 Each term of the equation is now divided by the coefficient with the largest absolute magnitude, 1337.

$$0.000749x^5 + 0.00719x^4 + 0.0135x^3 + 0.150x^2 + x - 0.425 = 0 \quad (4)$$

STEP 3 The coefficients of Equation (4) are now set into the computer. Since the position of the switch for a negative coefficient is only relative to that of a positive coefficient, in this example a switch in the left position is chosen as positive, as shown in Figure 2. The various coefficient potentiometers are set at their various percentages of full scale:

x^5 dial set at 0.000749 --- switch to left
x^4 dial set at 0.00719 --- switch to left
x^3 dial set at 0.0135 --- switch to left
x^2 dial set at 0.150 --- switch to left
x dial set at 1.000 --- switch to left
x^0 dial set at 0.425 --- switch to right

STEP 4 The range of the root dial is now from 0 to 1.0. This range is explored and it is found that when this dial indicates 0.40 a zero reading is obtained on the voltmeter. Therefore, 0.40 is a real root of the equation.

STEP 5 The range of the root dial is now changed to 0 to -1.0 by reversing the existing switch positions of the odd power terms. Thus the switches for the x^5, x^3, and x terms are thrown to the right. (The constant, or x^0 term, is considered an even power.) This region is explored but no roots are found.

STEP 6 Equation (3) is now transformed into an equation whose roots are 1/10 the roots of Equation (3). Let $k = 1/10$,

$$x^5 + (9.6)kx^4 + (18)k^2x^3 + (201.2)k^3x^2 + (1337)k^4x - k^5(568.4) = 0$$

$$x^5 + \frac{9.6}{10}x^4 + \frac{18}{100}x^3 + \frac{201.2}{1000}x^2 + \frac{1337}{10000}x - \frac{568.4}{100000} = 0$$

$$x^5 + 0.96x^4 + 0.18x^3 + 0.2012x^2 + 0.1337x - 0.005684 = 0 \quad (5)$$

If any of the coefficients in Equation (5) had had an absolute magnitude larger than unity, each term of the equation would have been divided by this magnitude, as in Step 2. The switch positions as set in Step 5 with the coefficients of Equation (5) set on the coefficient dials make the range of the root dial from 0 to -10. This region is explored, and a second root is found to be -7.0.

STEP 7 The switch positions of the odd power terms are again reversed to explore the negative of the region as defined in Step 6. The range is now from 0 to 10. No additional roots are found.

STEP 8 Steps 6 and 7 are repeated for increasing sub-multiples of 10. The equations obtained are as follows:

When $k = 1/100$, Equation (2) becomes

$$x^5 + 9.6(10^{-2})x^4 + 1.8(10^{-3})x^3 + 2.012(10^{-4})x^2 + 1.337(10^{-5})x - 5.684(10^{-8}) = 0 \quad (6)$$

and the range of the root dial is from 0 to ± 100.

When $k = 1/1000$, Equation (2) becomes

$$x^5 + 9.6(10^{-3})x^4 + 1.8(10^{-5})x^3 + 2.012(10^{-7})x^2 + 1.337(10^{-9})x - 5.684(10^{-13}) = 0 \qquad (7)$$

and the range of the root dial is from 0 to ± 1000. No additional roots are found in these regions. Condition (a) has not been satisfied since only two roots have been found and Equation (2) is of the fifth degree, indicating that three roots have not been found. Condition (b), however, has been satisfied since ± 1000 is greater than the upper and lower limits of the real roots as found in Step 1.

STEP 9 Equation (2) must be tested for multiple roots. Following the procedure outlined in Appendix III, $f(x)$ and $f'(x)$ become

$$f(x) = x^5 + 9.6x^4 + 18x^3 + 201.2x^2 + 1337x - 568.4 = 0 \qquad (8)$$

$$f'(x) = 5x^4 + 38.4x^3 + 54x^2 + 402.4x + 1337 = 0 \qquad (9)$$

and the greatest common divisor is found to be $x + 7$. Therefore, -7.0 is a double root of Equation (2). The real roots of Equation (2) are 0.4, -7, -7. If desired, the remaining pair of complex roots may be obtained by solving the depressed equation. Expanding the factors formed of the roots already found, Equation (10) is obtained,

$$(x - 0.4)(x + 7)^2 = x^3 + 13.6x^2 + 43.4x - 19.6 = 0 \qquad (10)$$

Dividing Equation (2) by Equation (10), the depressed equation becomes

$$x^2 - 4x + 29 = 0 \qquad (11)$$

Solving Equation (11) by the quadratic formula,

$$x = \frac{4 \pm \sqrt{16 - 116}}{2} = 2 \pm j5$$

the roots of Equation (2) are found to be

$$0.4,\ -7,\ -7,\ 2 + j5,\ 2 - j5$$

STEP 10 Since all the roots have been found, this step is unnecessary.

THE DESIGN OF AN EXPERIMENTAL MODEL TO SOLVE QUADRATIC EQUATIONS

The first consideration in the design of an actual circuit based upon the arrangement in Figure 2 was that part of the circuit used to obtain the various powers of the independent variable. Upon examination of Figure 1, it is seen that if the potentiometers are connected in tandem without some form of isolating network, each one loads the preceding one. This loading effect can be made negligible provided the resistance of each potentiometer is large compared to the preceding one. This method is not very practicable since it leads to difficulty in obtaining potentiometers of the required resistance and all having the same physical size so that they may be conveniently ganged together.

The solution decided upon was to use an amplifier between each potentiometer. The amplifier serves only as an isolating network, and necessarily has a gain of unity. By using type 6SN7 tubes, two such amplifiers may be obtained from the same tube envelope, thereby leading to compactness. The transformers available had a turns ratio of 1:1, with center-tapped primaries and secondaries. By using one half the turns on the primary in the plate-cathode circuit, to which the output circuit is coupled, and using all the turns of the primary in a negative feedback path, the autotransformer action thus obtained is used to advantage. With an adjustable resistance, R, in the feedback path, the gain may be varied both above and below unity. This is desirable, since the magnitudes of the outputs of each channel must all be adjusted to the same maximum value. The complete amplifier circuit is shown in Figure 3. The capacitor across the secondary of the output transformer is necessary to adjust the overall phase shift of the channel to zero.

Since the output voltages representing each term may be either polarity and must be connected in series, isolating transformers are necessary in each channel. The a_n, or constant-term channel, contains only the coefficient potentiometer and the reversing switch, therefore the transformer is placed between these two elements. In order for the components of each term to appear as symmetrical as possible, the transformer in the x-term channel is placed immediately after the coefficient potentiometer and preceding the x potentiometer. This arrangement is shown in Figure 4. The same procedure is followed in the x^2-term channel, with the addition of a second transformer in the output circuit of the amplifier. A photograph of the computer is presented in Figure 5.

It is seen from Figure 2 that the maximum output voltage of the x-term channel is less than that of the constant channel, and that the maximum output voltage of the x^2-term channel is less than either of these. As shown in Figure 4, a fixed resistance of 20,000 ohms is placed across the secondary of the transformer in the constant channel. Since each of the

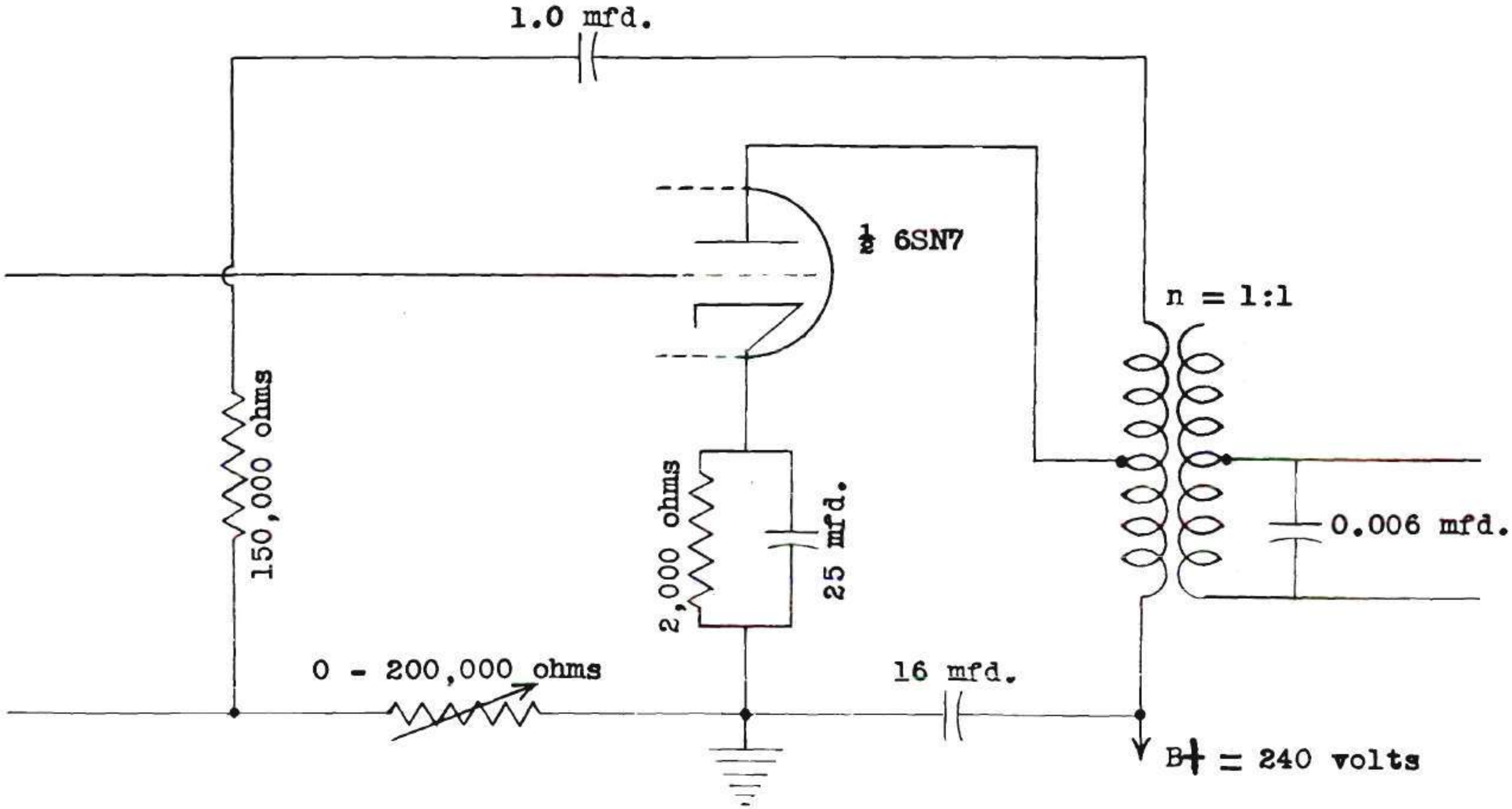


Figure 3

An isolating network consisting of an amplifier with a gain of unity and zero phase shift at the operating frequency.

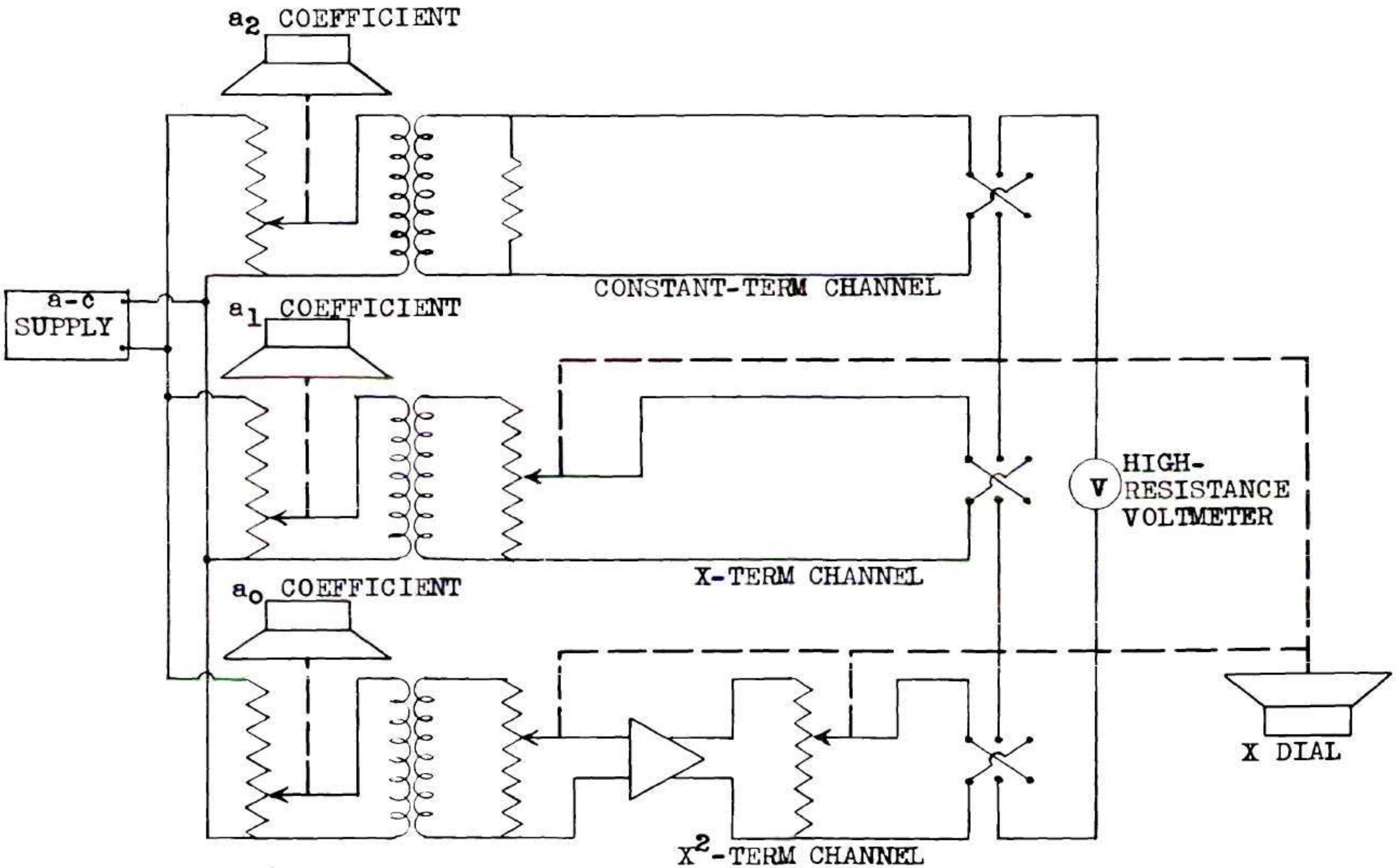
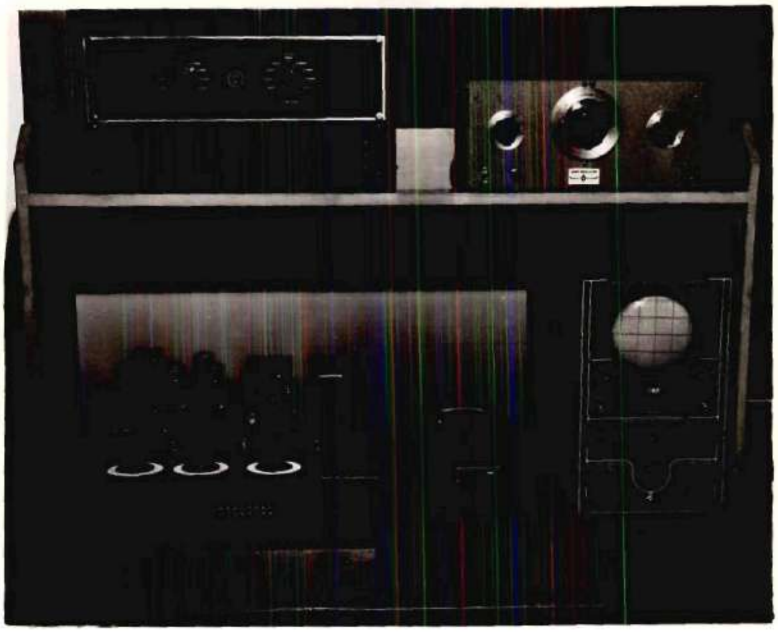
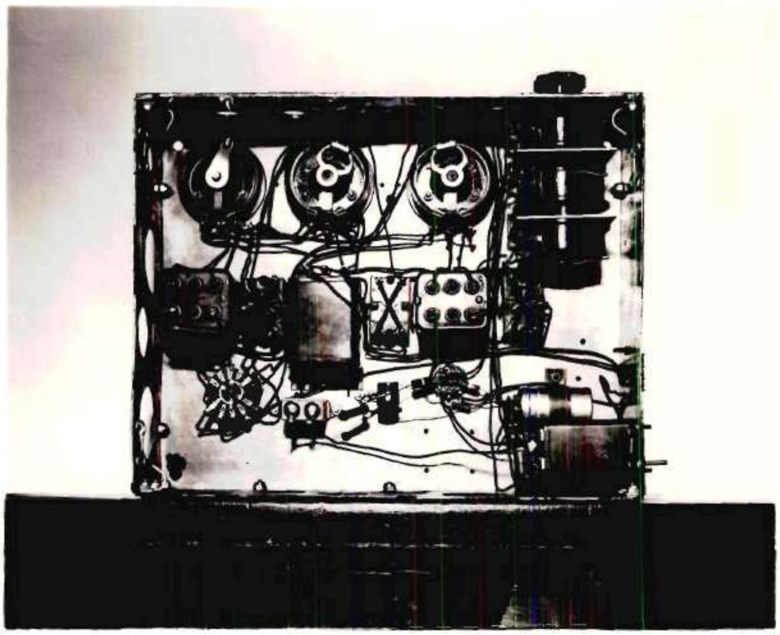


FIGURE 4

Diagram of an experimental computer to solve quadratic equations.



Quadratic Computer And Equipment



Bottom View Of Computer

Figure 5

potentiometers that are ganged on the common shaft also have a resistance of 20,000 ohms, it is seen that the maximum output voltages of the constant and x-term channels are now equal. The amplifier shown in Figure 3 is placed in the x^2-term channel, as shown in Figure 4. The maximum output voltage of the x^2-term channel can now be brought into agreement with the output voltages of the other two channels by adjusting the gain of the amplifier.

After the computer was constructed, the phase shift and magnitude of the output voltages of each channel were then checked separately with a vacuum-tube voltmeter and a cathode-ray oscilloscope. The voltages representing the constant term and the x-term agreed in both magnitude and phase as indicated on the oscilloscope. The magnitude of the x^2-term voltage was adjusted by varying the negative feedback in the amplifier. This was achieved by adjusting R. The phase shift of the x^2-channel voltage was adjusted to zero by placing a 0.006 mfd. capacitor across the secondary of the output transformer of the amplifier. The magnitudes of the channel voltages were checked on the oscilloscope by viewing, in phase opposition, the resultant of the maximum output voltages of each of the channels, taken two at a time. It was found that the transformers had a zero phase shift at a nominal frequency of 370 cycles, therefore, the phase shift of the amplifier was adjusted to zero at this frequency.

With all coefficient potentiometers and the root dial set at zero, it was found that the resultant output of the computer was not zero. By shorting each of the transformer secondaries one at a time, it was found that with the x^2 term completely shorted, the output could be reduced to zero. This indicated coupling between the transformers of the separate terms. Experimentation with two similar transformers showed that by placing their long axes mutually perpendicular the coupling was reduced to a negligible value. This modification was incorporated in the model with very good results. However, with the computer set to solve an equation, its output still showed the presence of hum and a considerable amount of distortion. By placing a bandpass filter at the output of the computer, the wave form was greatly improved, but the voltage was attenuated.

One dial plate, calibrated linearly from zero to ten, was obtained and installed underneath the x, or root, dial. Photostatic copies of this plate were obtained and used for the coefficient potentiometers. After their installation it was found that due either to a poor photostatic process or to a slight non-linearity of the potentiometers, or both, the dial reading did not check with a voltmeter placed across the potentiometer output. The voltmeter reading was assumed to be correct, and a correction curve was plotted for each coefficient dial. These data and curves are presented in Appendix V.

The accuracy of the x and x^2 channels was checked with a vacuum-tube voltmeter. The discrepancies found between the x and x^2 dial readings are probably due to non-linearities in the potentiometers or to a slight misalignment of the potentiometer arms on the common shaft. No attempt could be made to plot a correction curve for the root dial, since it involves the first and second powers of the variable and since the errors found for each were different. These errors do indicate, however, that the results obtained will be slightly inaccurate. These data are also presented in Appendix V.

Another large source of error in this computer is in the common potentiometer shaft. It was found that the Formica shaft, on which the potentiometer arms are mounted, does not remain rigid as it is rotated. This failure causes the potentiometer arms to be set at slightly different values of x for any setting on the dial. Likewise, two slightly different roots are indicated on the dial as the same null is approached from opposite directions.

There are three methods of indicating the null of the output voltage of the computer. The first method is to note the indication of a vacuum-tube voltmeter connected to the output terminals of the computer. The second method is to view the wave form of the computer output voltage on the oscilloscope. The third method is effected by connecting the computer output to the vertical deflection plates of the oscilloscope and the oscillator output to the horizontal plates.

The resulting pattern should theoretically be a straight line; but, due to hum and distortion in the computer output voltage, this pattern is a very narrow Lissajous figure. As the root dial is varied, the major axis of the figure is rotated about the center of the screen. When the major axis is horizontal, a root of the equation is read from the root dial. This method is much superior to either of the first two, especially when two roots lie very close together. Since the accuracy of the computer could be greatly increased by the use of this method, it was the one employed in solving the equations presented on later pages.

EXPERIMENTAL RESULTS

In order to check the validity of the roots obtained from the computer, a number of equations were solved. These equations have roots which cover the entire range of the root dial, in addition to having pairs of roots which are representative of the order of magnitudes normally encountered in quadratic equations. Since the scales on the coefficient dials were incorrect, it was found that the coefficient settings could be made much more quickly and accurately by using a voltmeter. A rotary switch was installed in the circuit in order that the same voltmeter could be used to separately set the coefficients and determine the roots. The oscilloscope was used in the manner previously described to obtain the roots of each of the equations presented here.

These equations were obtained by expanding factors which were chosen for their individual merit. The reduced equations and the transformed equations are not shown, but may be obtained by following the procedure previously outlined.

It is to be particularly noted that the computer consistently yields the value 0.08 for the root 0.1. This is partially explained by the correction curves for the x and x^2 terms. The equations having multiple roots were solved with exceptional accuracy, except for the one equation whose roots are 0.3, 0.3. For this particular equation the computer yielded two distinct roots placed symmetrically around the correct multiple root. However, with a slight change in the constant term of this equation, as indicated, the roots were made to converge on the correct multiple root.

For roots of the same sign and whose difference is 0.1, and which lie above about 0.4 and 0.5, great care is required to distinguish two distinct roots. However, if the roots are of opposite sign, the computer yields accurate results regardless of their absolute magnitude. The inability to distinguish between two roots of approximately the same magnitude and of the same sign is due to the potentiometers used. This ambiguity can be reduced by employing multi-turn potentiometers. The errors in the coefficient potentiometers may be reduced in the same manner, since for some equations, the coefficients could not be set accurately due to the coarseness of the windings.

As has been stated previously, the range of the roots that can be handled by the computer is practically unlimited. Regardless of how small or how large in absolute magnitude the roots may be, the original equation may be transformed so that the roots of the transformed equation are brought into the range of 0 to 1.0.

If an equation having conjugate complex roots is attempted to be solved on the computer, it is found that the straight line on the oscilliscope will never become horizontal, as is the case with real roots. As the root dial is turned through its range, this straight line will approach the horizontal and will make the smallest positive or negative angle with the horizontal at the value of x which corresponds to the real part of the complex root. This fact may be used to determine whether the roots, which have not been found after exploring the range between the upper and lower limits of the real roots, are multiple roots or complex roots without applying the test for multiple roots.

TABLE I: Equations Solved to Check Accuracy of Computer

Selected Factors	Resulting Equations	Measured Roots	Remarks
$(x + 0.1)(x + 0.1)$	$= x^2 + 0.2x + 0.01 = 0$	(-0.08, -0.08)	
$(x + 0.2)(x + 0.2)$	$= x^2 + 0.4x + 0.04 = 0$	(-0.2, -0.2)	
$(x + 0.3)(x + 0.3)$	$= x^2 + 0.6x + 0.09 = 0$	(-0.23, -0.36)	with $a_n = 0.92$, $x = -0.3, -0.3$
$(x + 0.4)(x + 0.4)$	$= x^2 + 0.8x + 0.16 = 0$	(-0.4, -0.4)	not quite horizontal
$(x + 0.5)(x + 0.5)$	$= x^2 + x + 0.25 = 0$	(-0.5, -0.5)	
$(x + 0.6)(x + 0.6)$	$= x^2 + 1.2x + 0.36 = 0$	(-0.6, -0.6)	
$(x + 0.7)(x + 0.7)$	$= x^2 + 1.4x + 0.49 = 0$	(-0.7, -0.7)	
$(x + 0.8)(x + 0.8)$	$= x^2 + 1.6x + 0.64 = 0$	(-0.8, -0.8)	
$(x + 0.9)(x + 0.9)$	$= x^2 + 1.8x + 0.81 = 0$	(-0.9, -0.9)	
$(x + 1)(x + 1)$	$= x^2 + 2x + 1 = 0$	(-1.0, -1.0)	
$(x + 0.1)(x - 0.1)$	$= x^2 - 0.01 = 0$	(0.08, -0.08)	
$(x - 0.2)(x + 0.3)$	$= x^2 + 0.1x - 0.06 = 0$	(0.195, -0.3)	
$(x + 0.2)(x + 0.3)$	$= x^2 + 0.5x + 0.06 = 0$	(-0.18, -0.3)	
$(x - 0.3)(x + 0.4)$	$= x^2 + 0.1x - 0.12 = 0$	(0.3, -0.4)	
$(x + 0.3)(x + 0.4)$	$= x^2 + 0.7x + 0.12 = 0$	(-0.3, -0.4)	
$(x - 0.4)(x + 0.5)$	$= x^2 + 0.1x - 0.20 = 0$	(0.4, -0.5)	
$(x + 0.4)(x + 0.5)$	$= x^2 + 0.9x + 0.20 = 0$	(-0.4, -0.5)	
$(x - 0.5)(x + 0.6)$	$= x^2 + 0.1x - 0.30 = 0$	(0.5, -0.6)	
$(x + 0.5)(x + 0.6)$	$= x^2 + 1.1x + 0.30 = 0$	(-0.5, -0.6)	
$(x - 0.6)(x + 0.7)$	$= x^2 + 0.1x - 0.42 = 0$	(0.6, -0.705)	
$(x + 0.6)(x + 0.7)$	$= x^2 + 1.3x + 0.42 = 0$	(-0.6, -0.7)	
$(x - 0.7)(x + 0.8)$	$= x^2 + 0.1x - 0.56 = 0$	(0.705, -0.805)	
$(x + 0.7)(x + 0.8)$	$= x^2 + 1.5x + 0.56 = 0$	(-0.7, -0.8)	
$(x - 0.8)(x + 0.9)$	$= x^2 + 0.1x - 0.72 = 0$	(0.8, -0.905)	
$(x + 0.8)(x + 0.9)$	$= x^2 + 1.7x + 0.72 = 0$	(-0.8, -0.9)	very distinct if roots are squared as explained in Appendix V

(Continued on page 29)

TABLE I: Equations Solved to Check Accuracy of Computer (continued)

Selected Factor	Resulting Equation	Measured Roots	Remarks
$(x - 0.9)(x + 1)$	$= x^2 + 0.1x - 0.9 = 0$	(0.9, -1.0)	when $k = 1/10$, $x = -0.8$
$(x + 0.9)(x + 1)$	$= x^2 + 1.9x + 0.9 = 0$	(-0.9, -1.0)	
$(x + 0.7)(x + 0.9)$	$= x^2 + 1.6x + 0.63 = 0$	(-0.72, -0.9)	
$(x + 0.6)(x + 0.8)$	$= x^2 + 1.4x + 0.48 = 0$	(-0.62, -0.81)	
$(x + 0.5)(x + 0.7)$	$= x^2 + 1.2x + 0.35 = 0$	(-0.52, -0.7)	
$(x + 0.4)(x + 0.6)$	$= x^2 + x + 0.24 = 0$	(-0.4, -0.6)	
$(x + 0.3)(x + 0.5)$	$= x^2 + 0.8x + 0.15 = 0$	(-0.3, -0.49)	
$(x + 0.2)(x + 0.4)$	$= x^2 + 0.6x + 0.08 = 0$	(-0.18, -0.41)	
$(x + 0.1)(x + 0.3)$	$= x^2 + 0.4x + 0.03 = 0$	(-0.08, -0.3)	
$(x + 0.6)(x + 0.9)$	$= x^2 + 1.5x + 0.54 = 0$	(-0.6, -0.9)	
$(x + 0.5)(x + 0.9)$	$= x^2 + 1.4x + 0.45 = 0$	(-0.505, -0.905)	
$(x + 0.4)(x + 0.9)$	$= x^2 + 1.3x + 0.36 = 0$	(-0.4, -0.92)	
$(x + 0.3)(x + 0.9)$	$= x^2 + 1.2x + 0.27 = 0$	(-0.3, -0.92)	
$(x + 0.2)(x + 0.9)$	$= x^2 + 1.1x + 0.18 = 0$	(-0.2, -0.9)	
$(x + 0.1)(x + 0.2)$	$= x^2 + 0.3x + 0.02 = 0$	(-0.08, -0.2)	
$(x + 0.1)(x + 0.5)$	$= x^2 + 0.6x + 0.05 = 0$	(-0.08, -0.5)	
$(x + 0.1)(x + 0.9)$	$= x^2 + x + 0.09 = 0$	(-0.08, -0.9)	
$(x + 0.1)(x + 1)$	$= x^2 + 1.1x + 0.1 = 0$	(-0.08, -0.8)	when $k = 1/10$, $x = -0.8$
$(x + 0.1)(x + 5)$	$= x^2 + 5.1x + 0.5 = 0$	(-0.08, -5.0)	
$(x + 0.1)(x + 9)$	$= x^2 + 9.1x + 0.9 = 0$	(-0.08, -9.0)	
$(x + 0.1)(x + 50)$	$= x^2 + 50.1x + 5 = 0$	(-0.08, -50.0)	
$(x + 0.1)(x + 90)$	$= x^2 + 90.1x + 9 = 0$	(-0.08, -90.5)	
$(x + 0.1)(x + 500)$	$= x^2 + 500.1x + 50 = 0$	(-0.08, -500)	
$(x + 0.1)(x + 5000)$	$= x^2 + 5000.1x + 500 = 0$	(-0.08, -5000)	
$(x + 0.01)(x + 4)$	$= x^2 + 4.01x + 0.04 = 0$	(-0.008, -4.0)	using $k = 10$ and $k = 1/10$
$(x + 0.01)(x - 4)$	$= x^2 - 3.99x - 0.04 = 0$	(-0.008, 4.0)	using $k = 10$ and $k = 1/10$

DESIGN CONSIDERATIONS

In the construction of a computer of the type described, great care must be exercised to eliminate coupling between any of the elements, especially between elements functioning in different terms of the equation. It is suggested that well-shielded transformers be used. The voltages induced in the circuit through stray coupling are usually shifted in phase with respect to the desired voltages. It can be easily seen that if this coupling is present the output voltage of the computer can never be adjusted to zero by changing only the magnitude of the independent variable.

The components used in the construction of the computer were found to cause a large change in phase shift with a small change of frequency. For this reason it is suggested that once the optimum operating frequency has been determined, an oscillator be incorporated in the computer. If desired, a high impedance, band-pass filter might be included at the output of the computer. The filter should preferably possess a zero phase shift characteristic at the operating frequency.

The ultimate accuracy of a computer using linear potentiometers is almost solely determined by the accuracy of the individual potentiometers used in the circuit. It is suggested that in the future, such designs as this should make use of extremely linear, multi-turn potentiometers, for example, those manufactured by Thomas B. Gibbs and Company. Specifications of the MICROPOT are reproduced in Appendix VII.

The circuit of Figure 4 can be modified so that an actual graph of the equation may be obtained. If the output of the computer, which is the sum of the terms of the equation, is applied to the vertical deflecting plates of an oscilloscope; and the input voltage applied to the horizontal plates; and if the x dial is rotated at a constant speed, the resulting trace on the scope will represent $y = f(x)$. The shaft of the ganged potentiometers could be connected to a small motor, which would continuously vary the independent variable through the range which had been previously selected. If this range is large, say from 0 to 1000, the trace represents $y = f(x)$ between $0 \leqq x \leqq 1000$. If it is desired to closely inspect the region between 0 and 10, or 0 and 1, etc., this can be done by choosing the proper value of k, as previously explained. Conversely, if a graph is given whose equation it is desired to find, it could be applied to the face of the oscilloscope, and the coefficients adjusted until the graph and the trace coincide. The desired equation could then be read from the coefficient settings. By a proper electronic switching arrangement the positive and negative regions between the same numerical limits could be observed simultaneously.

With some modifications the circuit of Figure 4 can be made capable of yielding complex roots. To accomplish this the independent variable must be varied in both phase and magnitude. By inserting a suitable 360° phase shifting device in each of the terms in the circuit of Figure 4 immediately following the coefficient potentiometers, and by ganging these phase shifters properly complex roots may be obtained in polar form. The phase shifters should be ganged such that for θ° rotation of the shaft the output of the x term will be shifted in phase θ°, the x^2 term shifted in phase $(2)\theta^\circ$, etc. If the equation contains complex coefficients the phase-shifting device should be set to indicate this shift when the shaft is at its zero position. This modification would impose even stricter requirements on the phase qualities of the circuit than already mentioned.

SUMMARY AND CONCLUSIONS

A computer is proposed for the purpose of solving an algebraic equation of any degree for its real roots. This computer operates on an a-c supply and evades the use of specially manufactured non-linear potentiometers by employing only linear potentiometers. By the application of certain mathematical theorems the range of the roots which the computer is capable of yielding becomes unlimited. By applying an additional theorem the degree of multiplicity of a root may be determined. A process is also indicated for overcoming the difficulty encountered when the roots lie very close together. A systematic procedure is outlined to be followed when solving an equation with the computer.

To test the validity and accuracy of the theory, an experimental model was constructed capable of solving quadratic equations for their real roots. The successful operation of this computer proved that the solution of an algebraic equation of any degree, based on the same theory, could be obtained with equal success. Many quadratic equations were solved with exceptional accuracy. These equations possess roots whose orders of magnitude cover a very wide range. The overall accuracy of this model was surprising, considering the large tolerances of the elements used in its construction.

A method is suggested by which the computer could be altered for producing a graph of the equation being solved.

Conversely, if the graph of an equation is given, its equation could be determined by this same method. Modifications of the design are also suggested to allow complex roots to be found. The capabilities of the computer are thus extended to permit the solution of any degree algebraic equation, having either real or complex coefficients, for its real or complex roots.

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APPENDIX I

LIMITS OF THE REAL ROOTS

APPENDIX I

LIMITS OF THE REAL ROOTS[4]

AN UPPER BOUND: $1 + \left|\frac{a_k}{a_0}\right|$ is an upper bound where a_k is the coefficient having the greatest absolute value of all those that differ in sign with a_0. If no coefficient differs in sign from a_0 there can be no positive roots and zero is an upper bound to the real roots.

EXAMPLE: Consider the equation

$$2x^4 + 8x^3 - 7x^2 + 16x - 3 = 0$$

An upper bound is $1 + \frac{7}{2} = 4.5$

A LOWER BOUND TO THE POSITIVE ROOTS: If the given equation is transformed into an equation whose roots are the reciprocals of the roots of the original equation, the upper bound to the roots of the transformed equation will be greater than the reciprocals of the roots of the given equation. Therefore, its reciprocal will be less than any of the positive roots of the given equation. A lower bound to the positive roots is

$$\frac{|a_n|}{|a_n| + |a_k|}$$

where a_k is the coefficient having the largest absolute value of all those that differ in sign with a_n.

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[4]Harry Sohon, Engineering Mathematics (New York: D. Van Nostrand Company, Inc., 1944), pp. 98-100.

EXAMPLE: A lower bound to the real positive roots of the equation

$$2x^4 + 8x^3 + 16x^2 - 47x - 3 = 0$$

is

$$\frac{3}{3 + 16} = \frac{3}{19}$$

A LOWER BOUND TO THE REAL ROOTS: If the given equation is transformed into a new equation each of whose roots is the negative of the corresponding root of the given equation, an upper bound to the roots of the transformed equation will be greater than the negatives of the roots of the given equation; therefore, its negative will be less than the roots of the given equation. A lower bound to the real roots of the given equation is

$$-1 - \left|\frac{a_k}{a_0}\right|$$

where $a_k(-1)^k$ differs in sign with a_0 and is numerically the greatest coefficient that satisfies this requirement as to sign.

EXAMPLE: In the equation

$$2x^4 + 8x^3 - 7x^2 + 16x - 3 = 0$$

the coefficients $(-1)^k a_k$ are

$$2, -8, -7, -16, -3$$

A lower bound to the real roots of the given equation is

$$-1 - \frac{16}{2} = -9$$

AN UPPER BOUND TO THE NEGATIVE ROOTS: By combining the steps in the preceding sections we find an upper bound to the negative roots of the given equation to be

$$-\frac{|a_n|}{|a_n| + |a_k|}$$

where $a_k(-1)^k$ differs in sign with $a_n(-1)^n$ and is numerically the greatest coefficient that satisfies this requirement as to sign.

EXAMPLE: To find an upper bound to the negative roots of

$$2x^4 + 8x^3 - 7x^2 + 16x - 3 = 0$$

rewrite the coefficients as $(-1)^k\, a_k$ for convenience

$$2,\ -8,\ -7,\ -16,\ -3$$

An upper bound to the negative roots is

$$-\frac{3}{3 + 2} = -0.6$$

APPENDIX II

TRANSFORMATIONS

APPENDIX II

TRANSFORMATIONS[5]

THEOREM: To transform an equation of the nth degree into an equation each of whose roots is k times the corresponding root of the original equation we proceed as follows: Multiply by k, k^2, k^3, etc., the coefficients of the given equation beginning with that of x^{n-1}. If any power of x below x^n is missing, it must be considered as present with a zero coefficient.

EXAMPLE: Transform the equation

$$x^2 + 3x + 2 = 0$$

into an equation whose roots are 1/10 those of the original equation ($k = 1/10$).

$$x^2 + (k)3x + (k^2)2 = 0$$

$$x^2 + (\tfrac{1}{10})3x + (\tfrac{1}{100})2 = 0$$

$$x^2 + 0.3x + 0.02 = 0$$

$$100x^2 + 30x + 2 = 0$$

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[5] *Ibid.*, p. 94.

APPENDIX III

MULTIPLE ROOTS

APPENDIX III

MULTIPLE ROOTS[6]

THEOREM: Any multiple root of $f(x) = 0$ of multiplicity $m > 1$ is a root of $f'(x) = 0$ of multiplicity $m-1$. A simple root of $f(x) = 0$ is not a root of $f'(x) = 0$.

THEOREM: If $f(x) = 0$ and $f'(x) = 0$ have at least one common root, then $f(x)$ and $f'(x)$ have a greatest common divisor $G(x)$, which actually involves x. A root of $G(x) = 0$ of multiplicity $m-1$ is a multiple root of $f(x)$ of multiplicity m. Conversely, any multiple root of $f(x) = 0$ of multiplicity m ($m \geqq 2$) is a root of $G(x) = 0$ of multiplicity $m-1$. If $q(x)$ denotes the quotient of $f(x)$ by $G(x)$, the roots of $q(x) = 0$ coincide with the distinct roots of $f(x) = 0$.

EXAMPLE: In the following example it is necessary to find the greatest common divisor of $f(x)$ and $f'(x)$. The theorem relating this process will not be stated here, but may be found in Dickson.[6] In carrying out this process it is to be noted that any constant multiplier may be added or dropped from the divisor, dividend or quotient at any time during the process.

Test $x^3 - 2x^2 - 4x + 8 = 0$ for multiple roots.

$$f(x) = x^3 - 2x^2 - 4x + 8 = 0$$

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[6] L. E. Dickson, New First Course in the Theory of Equations (New York: John Wiley and Sons, Inc., 1939), pp. 67-8.

$$f'(x) = 3x^2 - 4x - 4 = 0$$

First, find the greatest common divisor: Divide $f(x)$ by $f'(x)$, but to avoid fractions, divide $3f(x)$ by $f'(x)$,

$$\begin{array}{r|l}
 & x - 1 \\ \hline
3x^2 - 4x - 4 & 3x^3 - 6x^2 - 12x + 24 \\
 & \underline{3x^3 - 4x^2 - 4x} \\
 & -2x^2 - 8x + 24 \\
 & -x^2 - 4x + 12 \\
 & -3x^2 - 12x + 36 \\
 & \underline{-3x^2 + 4x + 4} \\
 & -16x + 32 \\
 & -x + 2
\end{array}$$

Since the degree of the remainder is less than the degree of the divisor, cease this operation, and neglect the quotient. The present remainder and divisor become the new divisor and dividend, respectively, for a second division. This process is repeated until there is a zero remainder. The divisor which yields a zero remainder is the greatest common divisor of the two polynomials.

$$\begin{array}{r|l}
 & -3x - 2 \\ \hline
-x + 2 & 3x^2 - 4x - 4 \\
 & \underline{-3x^2 - 6x} \\
 & 2x - 4 \\
 & \underline{2x - 4}
\end{array}$$

Since this division has zero remainder, the term $(-x + 2)(-1) = (x - 2)$ is the greatest common divisor of $f(x)$ and $f'(x)$. The root of $G(x) = x - 2 = 0$ in this example is 2 and is of multiplicity 1. Thus 2 is a double root of $f(x) = 0$.

APPENDIX IV

A TRANSFORMATION TO SEPARATE ADJACENT ROOTS

APPENDIX IV

A TRANSFORMATION TO SEPARATE ADJACENT ROOTS

It is desired to find an equation where roots are the squares of the roots of the original equation. The following material is quoted from Sohon:[7]

"Consider the equation in standard form

$$x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \ldots a_{n-1}x + a_n = 0$$

Transpose all the terms containing odd powers of x to one side and we have

$$-\left[x^n + a_2x^{n-2} + a_4{}^{n-4} + \ldots\right] = a_1x^{n-1} + a_3x^{n-3} + \ldots$$

Now square both sides of the equation

$$x^{2n} + a_2{}^2x^{2n-4} + 2a_2x^{2n-2} + 2a_4x^{2n-4} + a_4{}^2x^{2n-8} + \ldots$$
$$= a_1{}^2x^{2n-2} + a_3{}^2x^{2n-6} + 2a_1a_5x^{2n-6} + 2a_1a_3x^{2n-4} + \ldots$$

Rearranging this, we have

$$x^{2n} - x^{2n-2}(a_1{}^2 - 2a_2) + x^{2n-4}(a_2{}^2 - 2a_1a_3 + 2a_4)$$
$$- x^{2n-6}(a_3{}^2 - 2a_2a_4 + 2a_1a_5 - 2a_6 + \ldots = 0$$

Now substitute $-y$ for x^2 and the equation becomes

$$y^n + y^{n-1}(a_1{}^2 - 2a_2) + y^{n-2}(a_2{}^2 - 2a_1a_3 + 2a_4) + \ldots = 0$$

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[7]Sohon, op. cit., p. 123.

"If we now tabulate the coefficients of the x-equation and place the coefficients of the y-equation under the corresponding coefficients of the x-equation, the general rule will become evident.

TABLE VI-5

x	1	a_1	a_2	a_3	a_4	a_5
	1	a_1^2	a_2^2	a_3^2	a_4^2	a_5^2
		$-2a_2$	$-2a_1a_3$	$-2a_2a_4$	$-2a_3a_5$	$-2a_4a_6$
y			$+2a_4$	$+2a_1a_5$	$+2a_2a_6$	$+2a_3a_7$
				$-2a_6$	$-2a_1a_7$	$-2a_2a_8$
					$+2a_8$	$+2a_1a_9$
						$-2a_{10}$

EXAMPLE: Consider the equation whose roots are -0.8 and -0.9,

$$x^2 + 1.7x + 0.72 = 0$$

Find the equation whose roots are the squares of the roots of the above equation. For convenience a table is prepared similar to the preceding one

x	1	1.7	0.72
y	1	1.45	0.5184

The desired equation then becomes

$$y^2 + 1.45y + 0.5184 = 0$$

and its roots are the squares of the roots of the original equation, viz. -0.64 and -0.81.

APPENDIX V

CORRECTION CURVES

APPENDIX V

Data for Plotting Correction Curves for Coefficient Potentiometers

Meter Readings

Dial Setting	a_2	Correction	a_1	Correction	a_0	Correction
0	.0153	-.0153	0	0	0	0
0.02	.0385	-.0185	.020	0	.0150	+.005
0.04	.0560	-.0160	.0410	-.001	.0331	+.0069
0.06	.0770	-.0170	.0605	-.0005	.0540	+.006
0.08	.0965	-.0165	.082	-.002	.0713	+.0087
0.10	.116	-.016	.100	0	.0910	+.009
0.12	.134	-.014	.122	-.002	.113	+.007
0.14	.154	-.014	.142	-.002	.137	+.003
0.16	.171	-.011	.160	0	.152	+.008
0.18	.191	-.011	.184	-.004	.172	+.008
0.20	.210	-.010	.205	-.005	.192	+.008
0.22	.231	-.011	.225	-.005	.212	+.008
0.24	.249	-.009	.245	-.005	.232	+.008
0.26	.270	-.010	.265	-.005	.252	+.008
0.28	.285	-.005	.288	-.008	.272	+.008
0.30	.310	-.010	.305	-.005	.292	+.008
0.32	.329	-.009	.329	-.009	.312	+.008
0.34	.345	-.005	.348	-.008	.332	+.008
0.36	.364	-.004	.369	-.009	.352	+.008
0.38	.385	-.005	.390	-.010	.372	+.008
0.40	.405	-.005	.410	-.010	.392	+.008
0.42	.421	-.001	.429	-.009	.412	+.008
0.44	.440	0	.448	-.008	.432	+.008
0.46	.459	+.001	.460	0	.450	+.010
0.48	.479	+.001	.490	-.010	.472	+.008
0.50	.499	+.001	.505	-.005	.492	+.008
0.52	.519	+.001	.525	-.005	.505	+.015
0.54	.540	0	.543	-.003	.530	+.010
0.56	.559	+.001	.562	-.002	.550	+.010
0.58	.579	+.001	.581	-.001	.565	+.015
0.60	.590	+.010	.600	0	.585	+.015
0.62	.605	+.015	.621	-.001	.600	+.020
0.64	.635	+.005	.644	-.004	.625	+.015
0.66	.659	+.001	.660	0	.645	+.015
0.68	.680	0	.680	0	.670	+.010
0.70	.695	+.005	.700	0	.685	+.015
0.72	.720	0	.720	0	.715	+.005
0.74	.739	+.001	.740	0	.735	+.005
0.76	.760	0	.760	0	.755	+.005

(continued on page 47)

APPENDIX V

Data for Plotting Correction Curves for Coefficient Potentiometers (concluded)

Dial Setting	Meter Readings a_2	Correction	a_1	Correction	a_0	Correction
0.78	.785	-.005	.780	0	.775	+.005
0.80	.805	-.005	.800	0	.795	+.005
0.82	.815	+.005	.820	0	.815	+.005
0.84	.839	+.001	.840	0	.840	0
0.86	.860	0	.860	0	.860	0
0.88	.880	0	.880	0	.880	0
0.90	.900	0	.900	0	.900	0
0.92	.920	0	.920	0	.920	0
0.94	.945	-.005	.940	0	.940	0
0.96	.965	-.005	.960	0	.960	0
0.98	.982	-.002	.980	0	.980	0
1.00	1.000	0	1.000	0	1.000	0

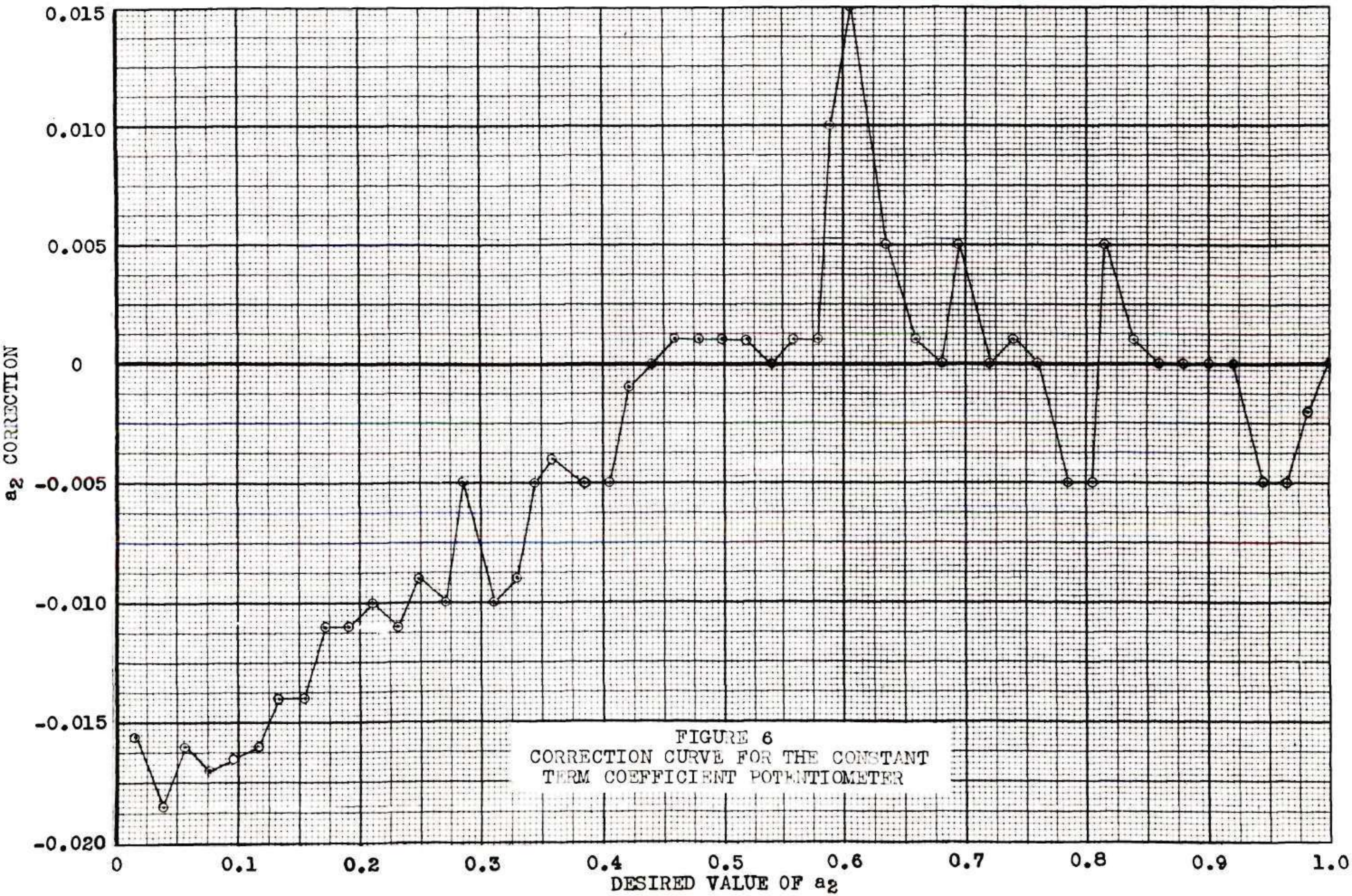


FIGURE 6
CORRECTION CURVE FOR THE CONSTANT TERM COEFFICIENT POTENTIOMETER

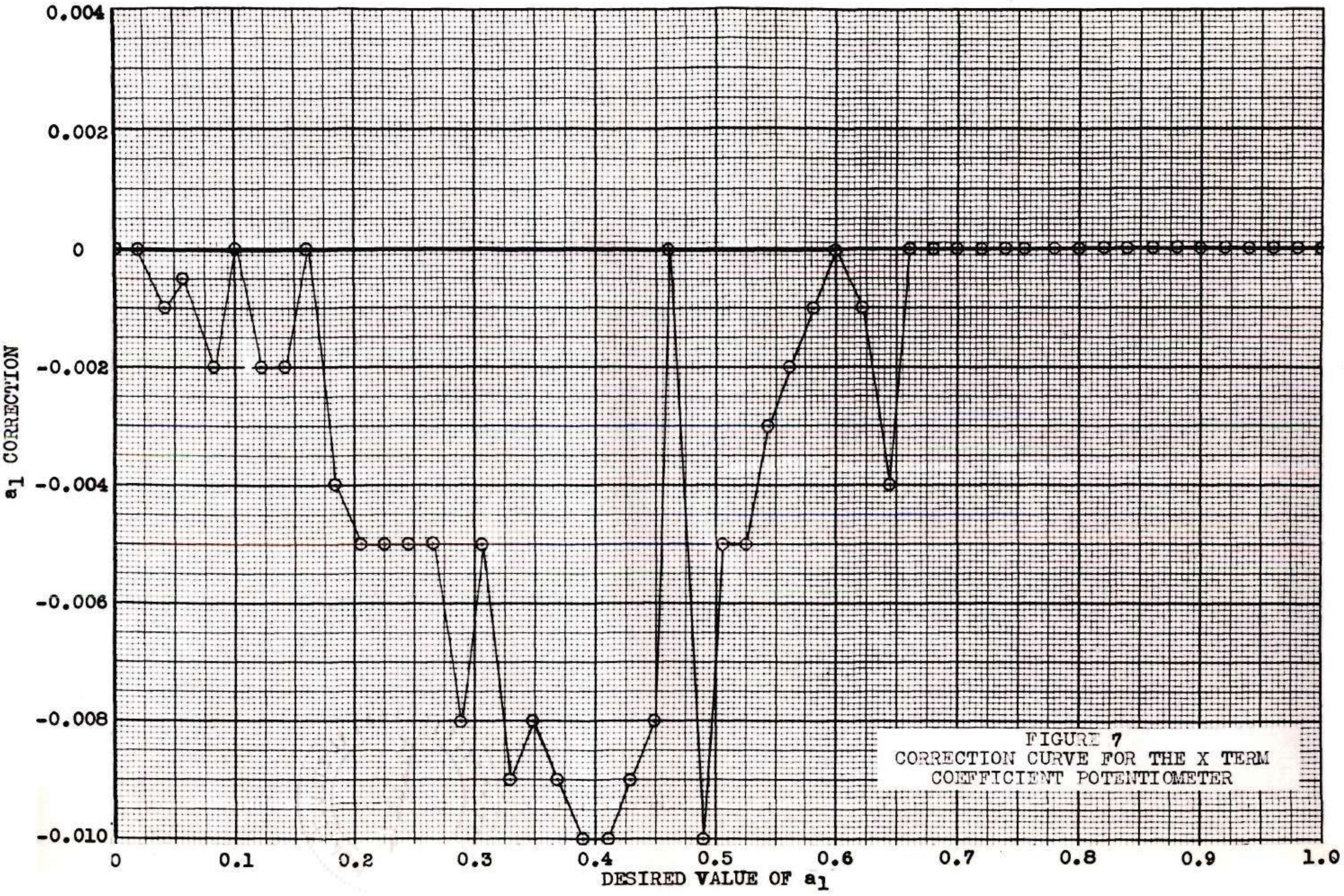


FIGURE 7
CORRECTION CURVE FOR THE X TERM
COEFFICIENT POTENTIOMETER

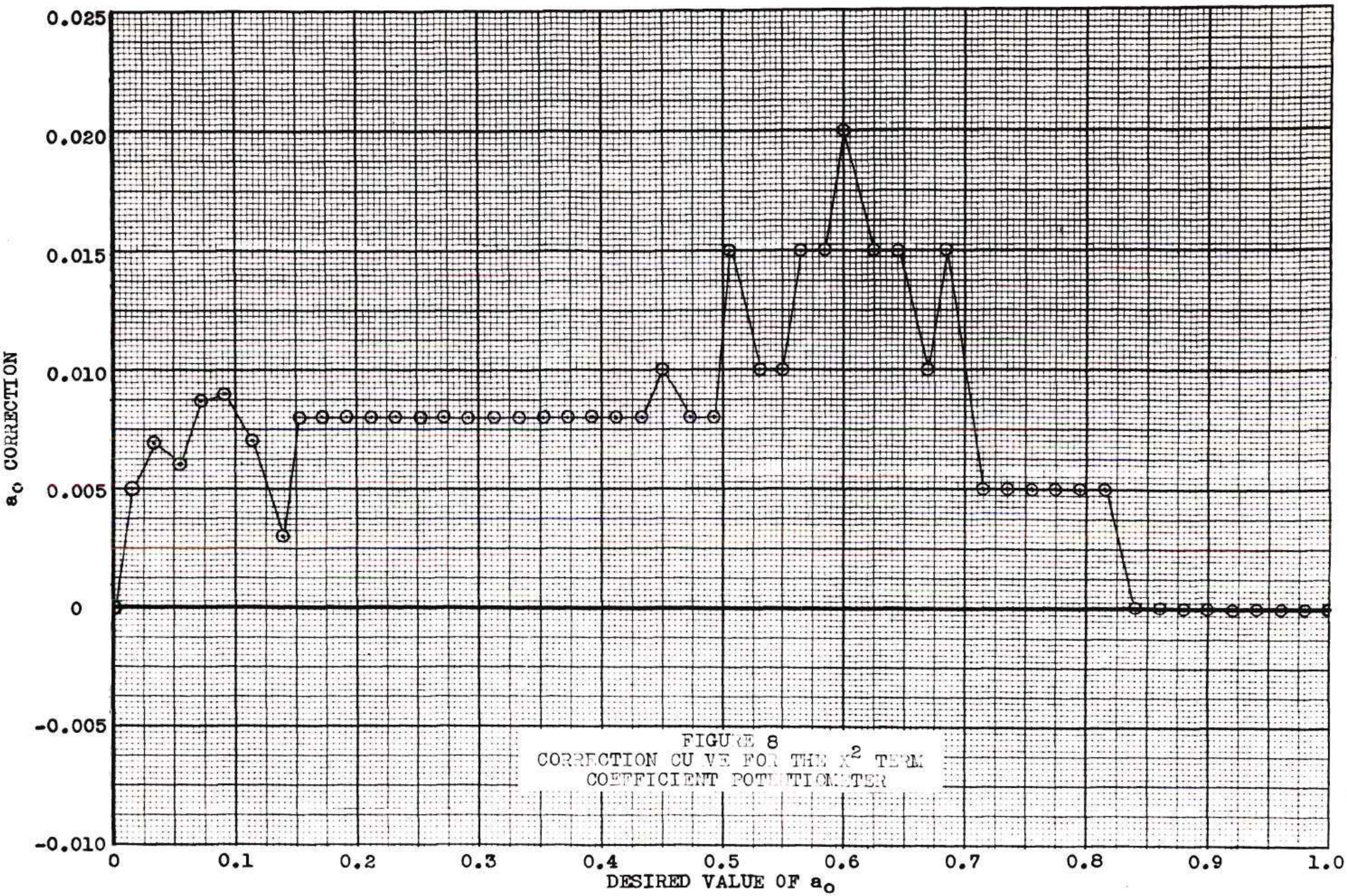


FIGURE 8
CORRECTION CURVE FOR THE X^2 TERM
COEFFICIENT POTENTIOMETER

APPENDIX V

Data for Plotting Correction in First and Second Powers of x

Root Dial Reading	Meter Reading at x Switch	Correction in x	Meter Reading at x^2 Switch	$\sqrt{\text{Meter Reading}}$	Correction in x^2
0	.0235	.0235	.00100	.03162	.03162
0.02	.0500	.0300	.00234	.04837	.02837
0.04	.0665	.0265	.00412	.06419	.02419
0.06	.0860	.0260	.00655	.08093	.02093
0.08	.100	.020	.0100	.1000	.0200
0.10	.117	.017	.0137	.1170	.0170
0.12	.138	.018	.0182	.1349	.0149
0.14	.154	.014	.0237	.1539	.0139
0.16	.173	.013	.0297	.1723	.0123
0.18	.192	.012	.0375	.1936	.0136
0.20	.211	.011	.0450	.2121	.0121
0.22	.230	.010	.0530	.2302	.0102
0.24	.249	.009	.0620	.2490	.0090
0.26	.269	.009	.0730	.2702	.0102
0.28	.288	.008	.0835	.2890	.0090
0.30	.304	.004	.0940	.3066	.0066
0.32	.322	.002	.104	.3225	.0025
0.34	.342	.002	.117	.3421	.0021
0.36	.362	.002	.130	.3606	.0006
0.38	.382	.002	.145	.3808	.0008
0.40	.400	0	.159	.3987	-.0013
0.42	.420	0	.176	.4195	-.0005
0.44	.440	0	.192	.4382	-.0018
0.46	.460	0	.212	.4604	.0004
0.48	.480	0	.229	.4785	-.0015
0.50	.500	0	.248	.4980	-.0020
0.52	.520	0	.268	.5177	-.0023
0.54	.540	0	.289	.5376	-.0024
0.56	.560	0	.309	.5559	-.0041
0.58	.580	0	.331	.5753	-.0047
0.60	.600	0	.357	.5975	-.0025
0.62	.620	0	.381	.6172	-.0028
0.64	.640	0	.408	.6387	-.0013
0.66	.660	0	.441	.6641	.0041
0.68	.680	0	.459	.6775	-.0025
0.70	.705	.005	.485	.6964	-.0036
0.72	.725	.005	.515	.7176	-.0024
0.74	.745	.005	.542	.7362	-.0038
0.76	.765	.005	.570	.7550	-.0050
0.78	.785	.005	.600	.7746	-.0054
0.80	.800	0	.638	.7987	-.0013

(continued on page 52)

APPENDIX V

Data for Plotting Correction in First and Second Powers of x (continued)

Root Dial Reading	Meter Reading at x Switch	Correction in x	Meter Reading at x^2 Switch	$\sqrt{\text{Meter Reading}}$	Correction in x^2
0.82	.825	.005	.665	.8155	-.0045
0.84	.845	.005	.700	.8367	-.0033
0.86	.865	.005	.738	.8591	-.0009
0.88	.882	.002	.775	.8803	.0003
0.90	.900	0	.805	.8972	-.0028
0.92	.925	.005	.840	.9165	-.0035
0.94	.945	.005	.880	.9381	-.0019
0.96	.965	.005	.920	.9592	-.0008
0.98	.990	.010	.960	.9798	-.0002
0.995	1.000	.005	1.000	1.000	.0050

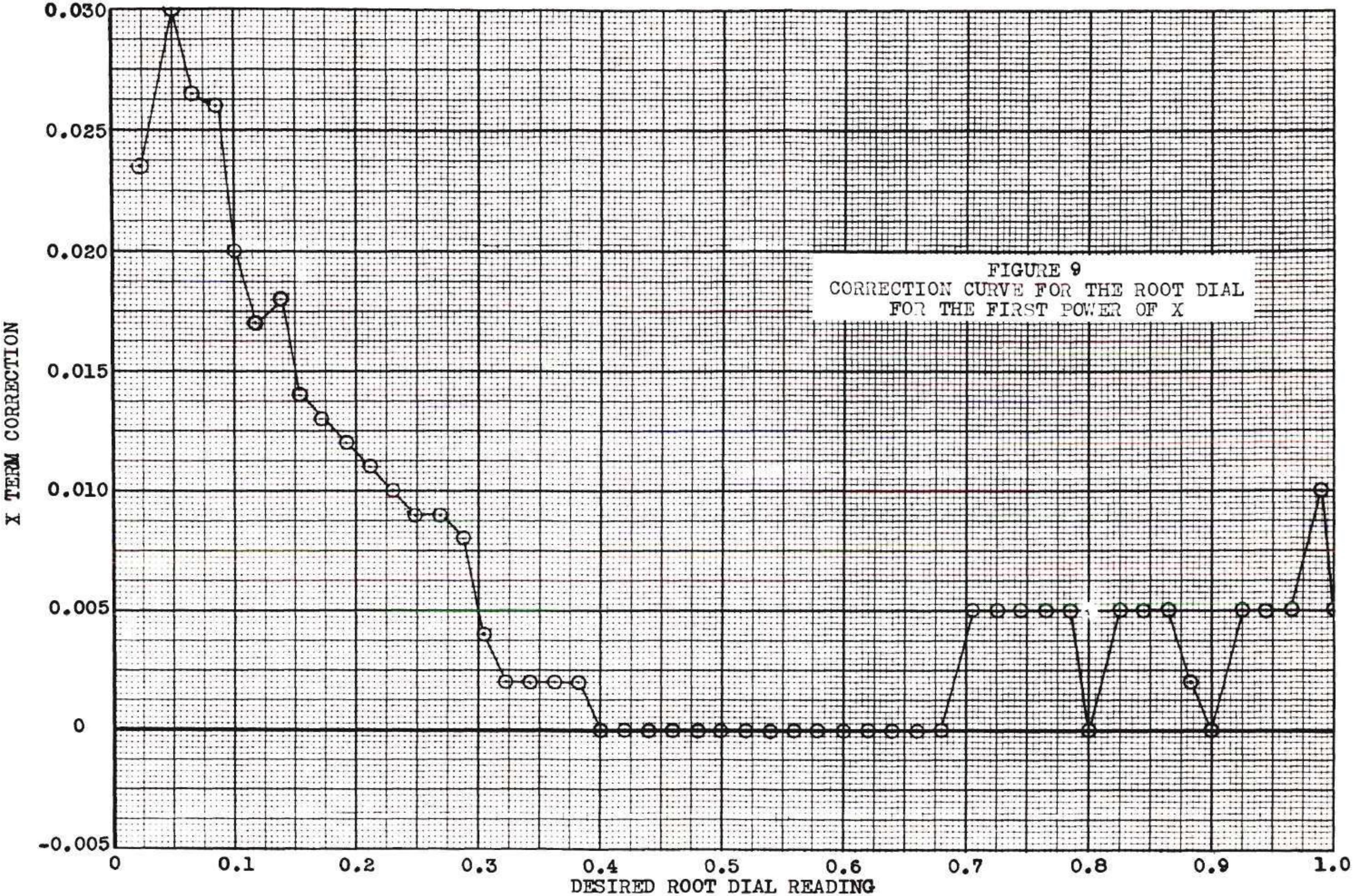


FIGURE 9
CORRECTION CURVE FOR THE ROOT DIAL
FOR THE FIRST POWER OF X
X TERM CORRECTION
0.030
0.025
0.020
0.015
0.010
0.005
0
-0.005
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
DESIRED ROOT DIAL READING

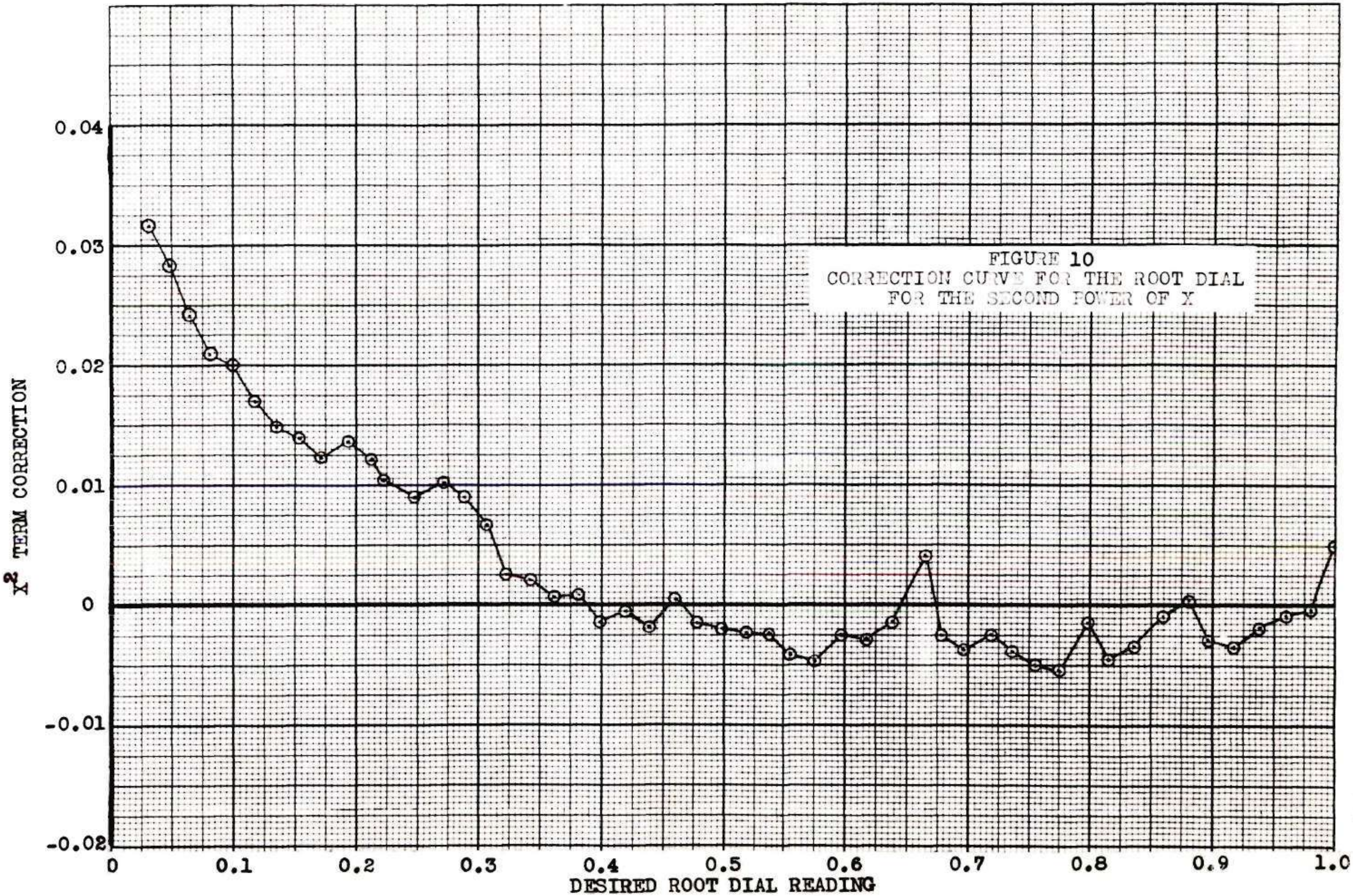


FIGURE 10
CORRECTION CURVE FOR THE ROOT DIAL
FOR THE SECOND POWER OF X

APPENDIX VI

MICROPOT SPECIFICATIONS

MICROPOT

Dec. 8, 1945

No. 31,002

THOMAS B. GIBBS & COMPANY

Division of the George W. Borg Corporation

TECHNICAL BULLETIN

Printed in U.S.A.

First order factors which influence the linearity of any potentiometer are variations in diameter of resistance wire, length per turn and spacing between turns. In the design of the MICROPOT, these quantities are chosen on the basis of average values, and while there are unavoidable errors in drawing or winding, these can be controlled to the point where linearity errors as small as .05% fall within the range of possibility.

Careful attention to other mechanical details further makes certain that linearity is built into the MICROPOT. And to insure its permanence, the entire resistance element (with soldered terminals attached) is pressure molded as an insert in the bakelite case. All MICROPOTS are tested for linearity at 41 rotor positions before leaving the factory.

RESOLUTION

In a wire wound potentiometer the moving contact must necessarily advance from one complete turn of resistance wire to the next; therefore, the change in resistance with rotation is not a truly continuous function but varies in small finite steps. The smaller these steps, the more accurately the potentiometer may be set. It is obvious that these steps (or resistance per turn of resistance wire) <u>must</u> be smaller than the linearity tolerance as defined above. These finite steps determine the <u>Resolution</u> (or resolving power) of the potentiometer and may be defined as the ratio of the resistance per turn (λ) to the total resistance at 100% rotation (R), or: $\Delta = \lambda / R$

Now the total resistance (R) is equal to the resistance per turn (λ) times the number of turns or: $\Delta = \lambda / R = \lambda / \lambda N = 1 / N$

In other words, the resolution of a potentiometer is dependent only on the number of turns.

The length of the winding core, therefore, becomes of paramount inportance when small resolution or high linearity and "resetability" accuracy is required. The spiral construction of the MICROPOT permits the use of a core $43\frac{1}{2}$" long in a $1\frac{3}{4}$" diameter case with the winding spread out over 3,600 degrees. (This core in a 281 degree potentiometer would be approximately $17\frac{3}{4}$ inches in diameter.)

TOTAL RESISTANCE

The circumference of the core (which determines the length per turn) and the specific resistivity of the wire directly affect the total resistance, but have no effect upon resolution or linearity.

The diameter of the resistance wire will have a direct effect upon the number of turns that can be wound on any given length core, and thus directly affects resolution. This indicates that small diameter resistance wire will give better resolution than a large diameter wire. The exact relationship between the various factors is beyond the scope of this bulletin.

In the MICROPOT the relation between length and circumference of the winding core has been so chosen that total resistances of from 1,000 to 100,000 ohms can be obtained while keeping the resolution such that linearity of less than one tenth of one percent can be held.

MICROPOT

Dec. 8, 1945

No. 31,002

THOMAS B. GIBBS & COMPANY

Division of the George W. Borg Corporation

TECHNICAL BULLETIN

Printed in U.S.A.

There are, in general, three characteristics of a linear, wire wound potentiometer which are of interest to the user. These are (1) Linearity, (2) Resolution, (3) Total Resistance. There are no theoretical limits between which any of the three characteristics must be held, but practical considerations tend to establish definite limits beyond which it is not desirable to go. It is the purpose of this bulletin to discuss these characteristics and their practical limits.

Throughout this discussion the following symbols will be used:

Θ = Percentage rotation
R_x = Resistance @ Θ percentage rotation
R = Resistance @ 100% rotation
λ = Resistance per turn
N = Number of turns to wind resistance @ 100% rotation
R_T = Total resistance between external terminals
Δ = Resolution

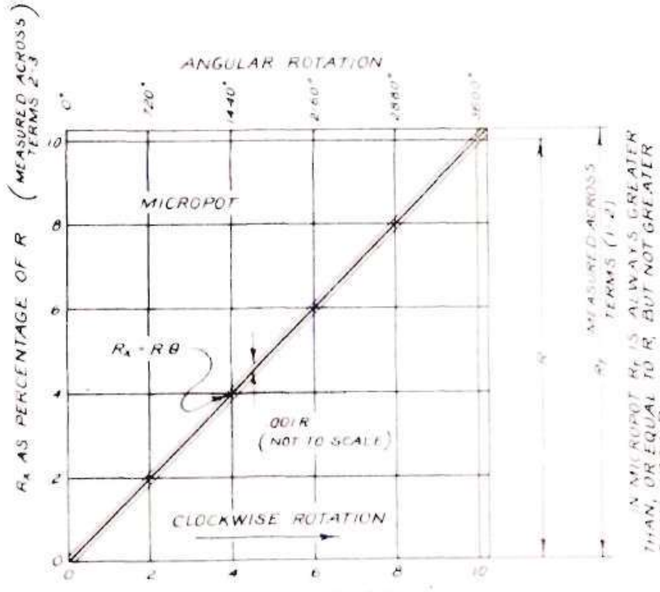
LINEARITY

A linear potentiometer can be defined as having percentage output equal to percentage rotation or $R_x = R\Theta$

Therefore, a potentiometer having .1% linearity can be defined by the equation

$$R_x = R\Theta \pm .001 R$$

This is the linearity expression used in defining the performance of the MICROPOT. Note that the error, expressed in ohms, is a constant once R is established. This definition may be shown graphically thus



Note that the MICROPOT linearity is carried right to the counter clockwise stop.

MICROPOT
Specifications

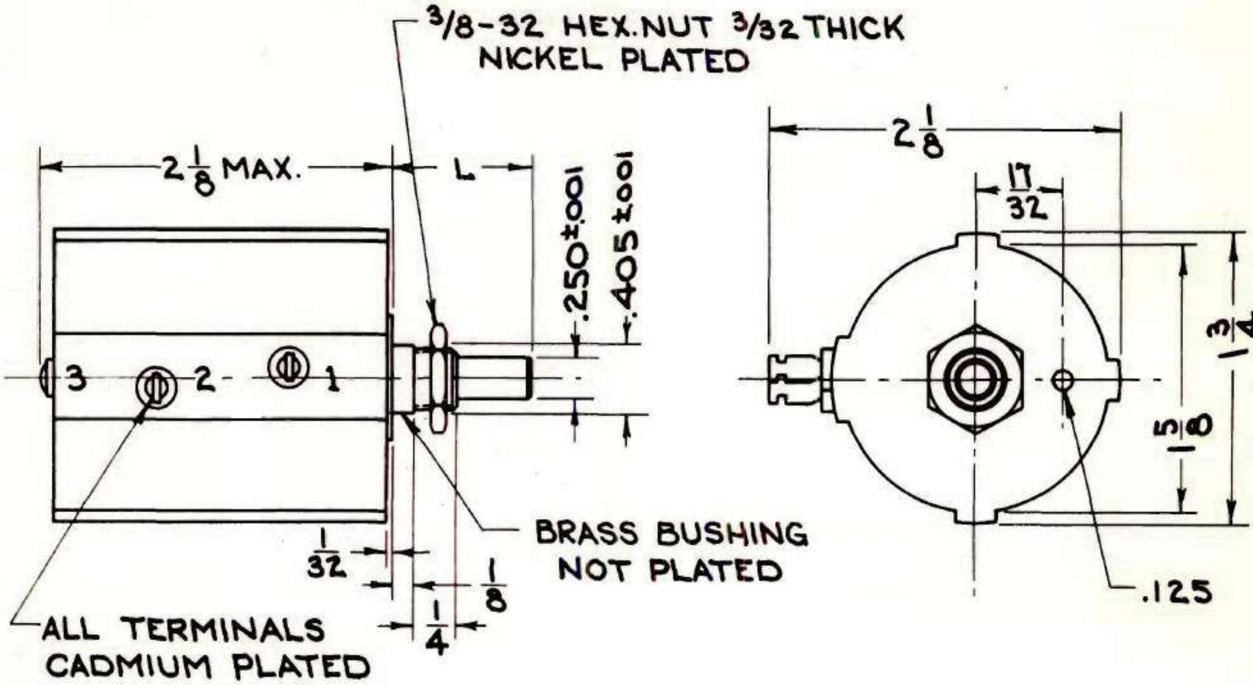
Dec. 8, 1945
No. 31,001

THOMAS B. GIBBS & COMPANY

Division of the George W. Borg Corporation

TECHNICAL BULLETIN

Printed in U.S.A.



SPECIFICATIONS

1.	Mechanical Rotation	—	3,600° +15° −0° (*)
2.	Electrical Rotation	—	3,600° +14.4° −0° (*)
3.	Total Resistance (± 5%)	—	1M, 2M, 5M, 10M, 20M and 30M (Stock Sizes)
4.	Linearity Accuracy	—	±0.1% (As defined in bulletin 31002)
5.	Torque	—	Not over 1.5 inch ounce running Not over 3.0 inch ounce starting
6.	Power Dissipation	—	2 Watts @ 71°C
7.	Shaft Extension "L"	—	7/8" or 1 5/8"
8.	Life	—	Not less than 50,000 cycles (1 million Revolutions)

(*) These tolerances are independent of each other.